Zero interface tension at the deconfining phase transition for a matrix model of a $SU(\infty)$ gauge theory

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Abstract

Using a matrix model, we model the deconfining phase transition at nonzero temperature for a $SU(N)$ gauge theory at large N. At infinite N the matrix model exhibits a Gross-Witten-Wadia transition. We show that as a consequence, both the order-disorder and the order-order interface tensions vanish identically at the critical temperature T_d. We estimate how these quantities vanish in the matrix model as $T \to T_d$ and as $N \to \infty$. The numerical solution of the matrix model suggests possible non-monotonic behavior in N for relatively small values of $N \sim 5$.

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I. INTRODUCTION

Because of dimensional transmutation, the properties of $SU(N)$ gauge theories are of great interest. Once one mass scale is set, all other properties are in principle determined. For example, if the theory exhibits a deconfining phase transition at a temperature T_d, then the nature of the phase transition is completely determined once the value of T_d is known.

Understanding the deconfining phase transition at small N requires numerical simulations on the lattice [1, 2]. Using these results, a matrix model was developed to model deconfinement. These models involve zero [3, 4], one [5], and two [6, 7] parameters. They are soluble analytically for two and three colors, and numerically for four or more colors.

This matrix model is also soluble in the limit of infinite N [8]. The phase transition at infinite N is exceptional, and can be termed a Gross-Witten-Wadia transition [9–11]. For the deconfining transition, such a transition was first seen to occur on a femto-sphere [12–16]. For such a transition, at infinite N it exhibits aspects of both first and second order phase transitions. It is of first order in that the latent heat is nonzero and proportional to $\sim N^2$. It is also first order in that the Polyakov loop jumps from 0 to $\frac{1}{2}$ at T_d. On the other hand, it is of second order in that several quantities, such as the specific heat, exhibit nontrivial critical exponents. Such an unusual transition only occurs at infinite N, as at finite N the transition is of first order.

Consider a phase transition which can be characterized by the change of a single field. If the transition is of first order, then at the transition temperature there are two degenerate minima, with a nonzero barrier between them. Then the order-disorder interface tension is nonzero, given by the probability to tunnel between the two minima. Conversely, if the transition is of second order, at the transition temperature the two minima coincide. Then there is no barrier to tunnel between them, and the order-disorder interface tension vanishes.

In a gauge theory asking about the potential is more subtle. In a matrix model of an $SU(N)$ gauge theory, the potential exists in $N-1$ dimensions (the number of commuting diagonal generators). A physical question is to ask how the interface tensions behave, especially at the phase transition. There are two such interface tensions. There is the usual order-disorder interface tension, which exists only at T_d. There are also order-order interface tensions [17–22] , which are directly related to the 't Hooft loops for $Z(N)$ charges [23, 24]. These exist for all $T \geq T_d$.

In this paper we consider the interface tensions in the matrix model near T_d. We find that in the matrix model, both the order-order interface tension and the order-disorder interface tensions, vanish identically at $T = T_d$ when $N = \infty$.

There is a simple heuristic explanation for our results. Consider the potential for the simplest Polyakov loop, that in the fundamental representation, $\ell_1 = (1/N)\mathrm{tr}\,\mathbf{L}$, where $\mathbf{L}$ is the thermal Wilson line. Then the Gross-Witten-Wadia transition occurs because at infinite N, the potential for ℓ_1 is completely flat between 0 and $\frac{1}{2}$. That is, at T_d there are two distinict minima, as expected for a first order transition. Nevertheless, because the potential is flat at $N = \infty$ between the two minima, there is no barrier to tunnel between them, and so the interface tensions vanish. Such a flat potential was first found for the Gross-Witten-Wadia transition on a femto-sphere [12–16].

As noted, in a gauge theory there are other degrees of freedom. For example, one can consider higher powers higher powers of $\mathbf{L}$, $\ell_j = \frac{1}{N}\mathrm{tr}\mathbf{L}^j$, for $j = 2\ldots N-1$. For the Gross-Witten-Wadia transition, though, all ℓ_j vanish at T_d when $j \geq 2$, which allows us to consider the potential as a function of a single variable, ℓ_1. We stress, however, that this is true only at T_d, and not for $T \neq T_d$. Indeed, while we estimate the behavior of the (order-order) interface tensions for $T > T_d$, we cannot solve the problem in full generality. This is because away from T_d, *all* ℓ_j contribute, and it is a much more difficult problem than at T_d.

The order-disorder interface tension has been measured through numerical simulations on the lattice at $N = 3$ by Beinlich, Karsch, and Peikert [25]. Lucini, Teper, and Wenger give results for the order-disorder interface tension for $N = 3$, 4, and 6, and use these to extrapolate to $N = \infty$ [26]. On general grounds the order-disorder interface tension should be proportional to $\sim N^2 T_d^2$, with a coefficient which is naturally of order one. Instead, Ref. [26] finds a very small coefficient at $N = \infty$, $\approx .014$. In the matrix model the coefficient $\sim N^2$ vanishes identically, and the true behavior is only $\sim N$. This is because in the matrix model, the height of the barrier between the two distinct minima is small, $\sim 1/N$. We discuss this further in Sec. (IV) and in the Conclusions, Sec. (V).

II.   REVIEW OF LARGE N THERMODYNAMICS

We are interested in the thermodynamics of pure $SU(N)$ gauge theory for temperatures a few times that for the deconfining phase transition at T_d. The order parameter of the

thermodynamics is taken to be the Wilson line:

$$\mathrm{L}(\vec{x}) = \mathrm{P}\exp\left(ig\int_0^\beta A_0(\vec{x},\tau)d\tau\right). \tag{1}$$

By a gauge transformation, we can diagonalize the field A_0 as

$$A_0^{ij} = \frac{2\pi T}{g}\,q_i\,\delta^{ij}, \tag{2}$$

where $i,j = 1\cdots N$ and the eigenvalues q_i are subject to the $SU(N)$ constraint $\sum_i^N q_i = 0$. In our model q_i are the fundamental variables to characterize the transition. We assume that after integrating out the other components of gluon field A_i, that we obtain an effective potential for q_i [3–8].

$$\tilde{V}_{eff}(q) = -d_1(T)\tilde{V}_1(q) + d_2(T)\tilde{V}_2(q), \tag{3}$$

$$\tilde{V}_n(q) = \sum_{i,j=1}^N |q_i - q_j|^n\,(1 - |q_i - q_j|)^n\,. \tag{4}$$

The potential includes both perturbative ($\tilde{V}_2$) and non-perturbative ($\tilde{V}_1$) contributions. The temperature dependent functions d_1 and d_2 are given by

$$d_1(T) = \frac{2\pi}{15}\,c_1\,T^2\,T_d^2,\ \ d_2(T) = \frac{2\pi}{3}\left(T^4 - c_2 T^2 T_d^2\right). \tag{5}$$

At tree level, the kinetic term is

$$\tilde{K}(q) = \frac{1}{2}\mathrm{tr}F_{\mu\nu}^2 = \left(\frac{2\pi T}{g}\right)^2\sum_{i=1}^N \left(\nabla q_i(x)\right)^2\,. \tag{6}$$

In a mean-field approximation, the kinetic term does not contribution for a spatially homogeneous states, and so it can be ignored for thermodynamic quantities. The kinetic term does enter in computing the interface tension in the following sections.

In the infinite N limit we introduce a continuous variable $x = \frac{i}{N}$. Labeling the eigenvalue $q_i \to q(x)$, we introduce the eigenvalue density

$$\rho(q) = \lim_{N\to\infty}\frac{1}{N}\sum_i^N \delta(q - q_i) = \int_0^1 dx\,\delta[q - q(x)] = \frac{dx}{dq}\,. \tag{7}$$

At finite N, the identities

$$\sum_i^N 1 = N,\ \ \sum_i^N q_i = 0 \tag{8}$$

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become, at infinite N,

$$\int dq \rho(q) = 1, \quad \int dq \rho(q) q = 0. \tag{9}$$

The potential is proportional to N^2,

$$\tilde{V}_n(q) = N^2 V_n(q) = N^2 \int dx \, dy \, |q(x) - q(y)|^n \left(1 - |q(x) - q(y)|\right)^n$$

$$= N^2 \int dq \, dq' \, \rho(q) \, \rho(q') \, |q - q'|^n \left(1 - |q - q'|\right)^n. \tag{10}$$

This representation transforms the potential into a polynomial in q.

The minimum of Eq. (10) was found in Ref. [8]. The solution is

$$\rho(q) = 1 + b \cos dq, \quad -q_0 < q < q_0, \tag{11a}$$

$$d = \sqrt{\frac{12 d_2}{d_1}}. \tag{11b}$$

$$\cot(dq_0) = \frac{d}{3}\left(\frac{1}{2} - q_0\right) - \frac{1}{d\left(1/2 - q_0\right)}, \tag{11c}$$

$$b^2 = \frac{d^4}{9}\left(\frac{1}{2} - q_0\right)^4 + \frac{d^2}{3}\left(\frac{1}{2} - q_0\right)^2 + 1. \tag{11d}$$

For $T > T_d$, $d > 2\pi$ and $q_0 < \frac{1}{2}$. The eigenvalues do not span the full range between $-\frac{1}{2}$ and $\frac{1}{2}$. The density is discontinuous at the end points $\rho(\pm q_0) > 0$. For $T = T_d^+$, $q_0 = \frac{1}{2}$ and $d = 2\pi$, the density is continuous for all values of q in $[-1/2, 1/2]$. In particulalr it vanishes at the end points $\rho(\pm q_0) = 0$. For $T < T_d$, the theory is in confined phase, with a uniform distribution of eigenvalues over the unit circle, Eq. (11a) with $q_0 = \frac{1}{2}$ and $b = 0$.

For the potential Eq. (10) and the eigenvalue distribution Eq. (11a), at T_d, $q_0 = \frac{1}{2}$, and the potential is independent of b. Changing b from 0 to 1 interpolates between confined and deconfined phase, but does not change the potential. Hence b is a zero mode of the potential corresponding to changing the overall shape of the distribution. This will play an important role in the construction of the interface. For short we call the change of shape related to b the b-mode.

It is also worth emphasizing that in the derivation of Ref. [8], we have assumed that the eigenvalue density is symmetric in q. We can obtain different distributions by applying an arbitrary Z_N transform to a given solution. A Z_N transform of charge k, $k = 1 \ldots (N - 1)$, is given by

$$q_1, q_2, \cdots q_N \rightarrow q_1 + \frac{k}{N}, \cdots q_{N-k} + \frac{k}{N}, q_{N-k+1} + \frac{k - N}{N}, \cdots q_N + \frac{k - N}{N}. \tag{12}$$

Assuming $q_1 \leq q_2 \leq \cdots \leq q_N$ and $|q_i - q_j| < 1$, we can relabel the eigenvalues such that they are in an increasing order:

$$q_1, q_2, \cdots q_N \quad \rightarrow q_{N-k+1} + \frac{k-N}{N}, \cdots q_N + \frac{k-N}{N}, q_1 + \frac{k}{N}, \cdots q_{N-k} + \frac{k}{N}. \qquad (13)$$

In the infinite N limit, the Z_N tranform takes the following form. Define the inverse function of $x(q) = \int_{-q_0}^{q} dq' \rho(q')$ as $Q(x)$,

$$q = Q(x) \rightarrow q = \begin{cases} Q(x + 1 - \Delta) - 1 + \Delta & 0 < x < \Delta \\ Q(x - \Delta) + \Delta & \Delta < x < 1 \end{cases}. \qquad (14)$$

Since the potential is invariant under Z_N transformations, smooth changes in Δ are another zero mode of the potential, which we call the shift mode. This is also relevant to the construction of the interface.

We stress that both the b and shift modes are become zero modes only at infinite N. The former is because of the flatness of the potential. The latter is because the Z_N symmetry becomes a continuous $U(1)$ at $N = \infty$. We comment on what happens at finite N later.

III. INTERFACE TENSION

An interface is a topological object interpolating between two vacua of the theory. Suppose the two vacua are separated in the z direction and extended in the $x - y$ plane. The effective action has an area law when the transverse size L_{tr} is large: $S_{eff} = \alpha L_{tr}$. The proportionality constant defines the interface tension. Up to cubic order in the perturbative expansion, the order-order interface tension exhibits Casimir scaling [19, 20]

$$\alpha \propto k(N - k). \qquad (15)$$

An important question to address is whether the order-order interface tension satisfies Casimir scaling in the matrix model.

An interface tension necessarily involves a spatial gradient along the z direction. Consequently, the kinetic term must be included. Since the potential is simple when written in terms of the eigenvalue density $\rho(q)$, it is useful to write the kinetic term in terms of the same variable as well. Assuming that there is a spatial gradient only along the z direction,

$$\tilde{K}(q) = N^2 \, K(q) = N^2 \frac{(2\pi T)^2}{g^2 N} \int dz dx \left(\frac{\partial q(x, z)}{\partial z} \right)^2, \qquad (16)$$

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where the partial derivative is taken at fixed x.

We start with the eigenvalue density $\rho(q, z) = \partial x(q, z)/\partial q$. Assuming that the range of the eigenvalue distribution does not change over the interface,

$$x(q, z) = \int_{-q0}^{q} dq' \rho(q', z). \tag{17}$$

Using the chain rule, we have

$$
\begin{aligned}
\frac{\partial q(x, z)}{\partial z} &= -\frac{\partial q(x, z)}{\partial x} \frac{\partial x(q, z)}{\partial z} \\
&= -\frac{\partial_z \int^q dq' \rho(q', z)}{\rho(q, z)}.
\end{aligned}
\tag{18}
$$

The kinetic term becomes (c.f. [27])

$$K(q) = \int dz dq \frac{\left(\partial_z \int^q dq' \rho(q', z)\right)^2}{\rho(q, z)}. \tag{19}$$

A. Interface tension at T_d

For simplicity consider the interface tension at T_d first. At the transition there are both order-order interface and order-disorder interfaces. Due to the complicated form of the kinetic energy Eq. (19), solving for the full solution of the interface seems to be hopeless. However, the presence of the zero modes allows us to show that both interface tensions vanish. It is straightforward to construct an interface using the shift mode:

$$
q = \begin{cases}
Q(x + 1 - \Delta f(z)) - 1 + \Delta f(z), & 0 < x < \Delta f(z); \\
Q(x - \Delta f(z)) + \Delta f(z), & \Delta f(z) < x < 1
\end{cases}
\tag{20}
$$

with $f(-L) = 0$ and $f(L) = 1$ at two boundaries of the interface. A k-wall interpolating two vacua related by Z_N^k transformation corresponds to $\Delta = \frac{k}{N}$.

The idea is to take $f(z) = \frac{z}{2L} + \frac{1}{2}$ such that $\frac{\partial q}{\partial z} \sim \frac{1}{L}$. As f is a zero mode, this not change the potential energy, while the kinetic energy is supressed by $\frac{1}{L}$. In the limit $L \to \infty$, action vanishes as $1/L$, and so the interface tension vanishes at T_d.

This does not work for the shift mode. To see that, we need to take a close look at the kinetic term. On the interface, the density of eigenvalues is given by

$$
\rho = \begin{cases}
1 + \cos 2\pi(q + 1 - \Delta f(z)), & Q(1 - \Delta f(z)) - 1 + \Delta f < q < -1/2 + \Delta f(z) \\
1 + \cos 2\pi(q - \Delta f(z)), & -1/2 + \Delta f(z) < q < Q(1 - \Delta f(z)) + \Delta f(z).
\end{cases}
\tag{21}
$$

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Integrating with respect to q, we obtain

$$x = q + \frac{1}{2} + \frac{1}{2\pi}\sin[2\pi(q - \Delta f)], \quad Q(1 - \Delta f) - 1 + \Delta f < q < Q(1 - \Delta f) + \Delta f, \quad (22)$$

from which it follows

$$\frac{\partial q}{\partial z} = -\frac{\cos[2\pi(q - \Delta f)]\Delta f'}{1 + \cos[2\pi(q - \Delta f)]}. \quad (23)$$

Plugging the gradient Eq. (23) into Eq. (19), we identify a non-integrable singularity at $q = 1/2 + \Delta f$. Therefore, we conclude that the interface built by the shift mode is ruled out by the divergent kinetic energy.

A second possibility is to build an interface with the b-mode. The two vacua at the ends of the interface are joined through a confining phase in the middle. Defining Q_b as the inverse function of

$$x = q + \frac{1}{2} + \frac{b}{2\pi}\sin(2\pi q). \quad (24)$$

At $T = T_d$, $Q_1(x)$ reduces to the distribution for the deconfined phase, and $Q_0(x)$ for the confined phase. The interface is constructed as

$$\text{part I} : q = Q_{b(z)}(x), \quad -L < z < 0 \leftrightarrow$$

$$\text{part II} : q = \begin{cases} Q_{b(z)}(x + 1 - \Delta) - 1 + \Delta, & 0 < x < \Delta; \\ Q_{b(z)}(x - \Delta) + \Delta, & \Delta < x < 1. \end{cases} \quad 0 < z < L, \quad (25)$$

with the boundary conditions $b(\pm L) = 1$, $b(0) = 0$. The distributions from part I and part II joining at $z = 0$ are identical and given explicitly by $q = x - 1/2$. We note that part II is a $Z(N)$ transform of part I, flipped in z. We will show below that the kinetic energy is not divergent, so that we can apply the previous argument to arrive at the suppression in $1/L$. Since the path given by the first and second lines of Eq. (25) are related by Z_N transformations, they necessarily have the same kinetic and potential energies. It is sufficient to restrict ourselves to the first line. It is easy to find the gradient

$$\frac{\partial q}{\partial z} = -\frac{b'}{2\pi}\frac{\sin 2\pi q}{1 + b\cos 2\pi q}, \quad (26)$$

where here and in the following primes denote the derivatives with respect to the argument. We see possible singularities at $q = \pm 1/2$ from the denominator are cancelled by the numerator, giving rise to a finite result for the kinetic energy. For a given b, it is not difficult

to compute the integral in q by contour integration,

$$\int_{-1/2}^{1/2} dq \rho(q) \left(\frac{\partial q}{\partial z}\right)^2 = \left(\frac{b'}{2\pi}\right)^2 \frac{-b^4 + b^2 \left(5 - 3\sqrt{1 - b^2}\right) + 4 \left(-1 + \sqrt{1 - b^2}\right)}{b^2 \sqrt{1 - b^2} \left(-1 + \sqrt{1 - b^2}\right)^2}. \qquad (27)$$

We checked that the kinetic term is finite in the limit $b \to 1$.

B.   Interface tension at $T > T_d$

The order-disorder interface tension is only defined at T_d. Above T_d, the order-order interface tension, equivalent to the 't Hooft loop, is nonzero. However, we were not able to compute the order-order interface tension in full generality.

Let's consider two limiting situations, $\Delta \to 0$ and $T \to T_d$. We consdier them in turn.

In the case $\Delta \to 0$, the end points of the interface are given by

$$q = Q(x), \quad q = \begin{cases} Q(x + 1 - \Delta) - 1 + \Delta, & 0 < x < \Delta; \\ Q(x - \Delta) + \Delta, & \Delta < x < 1. \end{cases} \qquad (28)$$

The eigenvalues between $\Delta < x < 1$ have change infintesimally, while those between $0 < x < \Delta$ have a finite jump; the latter, however, are suppressed because there are few of them. With this in mind, we write down the following path for the interface:

$$q = (Q(x + 1 - \Delta) - 1 + \Delta - Q(x)) \, g(z) + Q(x), \qquad 0 < x < \Delta; \qquad (29)$$

$$q = (Q(x - \Delta) + \Delta - Q(x)) \, f(z) + Q(x), \qquad \Delta < x < 1. \qquad (30)$$

The unknown functions $g(z)$ and $f(z)$ interpolate between 0 and 1. They could in principle depend on x, which characterizes the change of shape of the eigenvalue density. To leading order in Δ we can ignore this dependence. Now we can work out the potential energy along the path

$$V = \int_0^\Delta dx dy V \left(q(x) - q(y)\right) + \int_\Delta^1 dx dy V(q(x) - q(y)) + 2 \int_0^\Delta dx \int_\Delta^1 dy V(q(y) - q(x)). \qquad (31)$$

The first term is of order $O(\Delta^3)$ and may be ignored. The second term starts with the vacuum potential energy as that at leading order. The third term is of order $O(\Delta)$. We need to know the $O(\Delta)$ correction of the potential to the vacuum one:

$$\delta V = \int_0^1 dx dy \frac{\partial V}{\partial |q(x) - q(y)|} \Delta |Q'(x) - Q'(y)| f$$

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$$+ 2 \int_0^{\Delta} dx \int_0^1 dy \left(V(q(y) + q_0) - V(q(y) + q_0 + (1 - 2q_0)g) \right). \tag{32}$$

Evaluating Eq. (32) for the vacuum solution Eq. (11a), we find the first term vanishes identically, while the second term gives

$$\delta V = \frac{d^2}{12}(1 - 2q_0)^4 g^2 (1 - g)^2. \tag{33}$$

The kinetic energy reads

$$\int dz dx \left(\frac{\partial q}{\partial z} \right)^2 \tag{34}$$

$$\text{with } \frac{\partial q}{\partial z} = \begin{cases} (2q_0 - 1)g', & 0 < x < \Delta; \\ \Delta \left(1 - Q'(x) \right) f', & \Delta < x < 1. \end{cases} \tag{35}$$

We ignore the contribution from $\Delta < x < 1$ because it is of order $O(\Delta^2)$. Combining the kinetic and potential terms, we have

$$\int dz(K + \delta V) = \int dz \left(\Delta(1 - 2q_0)^2 g'^2 + \Delta \frac{d^2}{12}(1 - 2q_0)^4 g^2 (1 - g)^2 \right). \tag{36}$$

We need to minimize the above action with the boundary condition $g(-\infty) = 0$ and $g(\infty) = 1$. Using a trick (if $(x - y)^2 \geq 0$, $x^2 + y^2 \geq 2xy$), we can obtain the minimum without solving for g:

$$\int dz(K + \delta V) \geq \Delta \int dz 2 \sqrt{(1 - 2q_0)^2 g'^2 \frac{d^2}{12}(1 - 2q_0)^4 g^2 (1 - g)^2}$$

$$= \Delta \int dg g(1 - g) \frac{d(1 - 2q_0)^3}{\sqrt{3}} = \Delta \frac{d(1 - 2q_0)^3}{6\sqrt{3}}, \tag{37}$$

which leads to

$$\alpha = \Delta \frac{d}{6\sqrt{3}} (1 - 2q_0)^3. \tag{38}$$

Now we look at the other limiting case: $T \to T_d$. We expect the interface to mimic the $T = T_d$ case, $i.e.$ two vacua joined through a confined distribution. We consider the following simple path:

part I : $\rho = a(z) + b(z)b_0 \cos dq, \quad -q_0 < q < q_0,$

part II : $\rho = \begin{cases} \frac{1}{2q_0}, & -q_0 - (1 - \Delta)(1 - 2q_0)f(z) < q < -(1 - 2\Delta)q_0 - (1 - \Delta)(1 - 2q_0)f(z); \\ \frac{1}{2q_0}, & -(1 - 2\Delta)q_0 + \Delta(1 - 2q_0)g(z) < q < q_0 + \Delta(1 - 2q_0)g(z), \end{cases}$

part III : $\rho = \begin{cases} a(z) + b(z) \cos d(q + 1 - \Delta), & Q_b(1 - \Delta) - 1 + \Delta < q < q_0 - 1 + \Delta; \\ a(z) + b(z) \cos d(q + 1 - \Delta), & -q_0 + \Delta < q < Q_b(1 - \Delta) + \Delta, \end{cases} \tag{39}$

We have $-L < z < -\frac{L}{2}$ for part I, $-\frac{L}{2} < z < \frac{L}{2}$ for part II and $\frac{L}{2} < z < L$ for part III, with the following boundary conditions

$$b(\pm L) = 1, \ b(\pm L/2) = 0, \ f(-L/2) = 0, \ f(L/2) = 1. \tag{40}$$

In Eq.(39) q_0 and b_0 are determined by Eqs. (11c) and Eq. (11d), respectively. The normalization condition $\int dq \rho(q) = 1$ forces $a(z)q_0 + \frac{b(z)}{d} \sin dq_0 = 1/2$, while the tracelessness condition $\int dq \rho(q) q = 0$ sets $f(z) = g(z)$. Furthermore, we have the boundary conditions The interface is composed of three parts. Part I makes use of the b-mode to deform the eigenvalue distribution. At $T \to T_d$, b mode is nearly a zero mode, with minimum cost of potential energy. Part II is unique to the case $T > T_d$, with the motion of the eigenvalues suppressed by $(1 - 2q_0)$. Part II is just a Z_N transform of part I, flipped in z.

Let us look at part I first. The potential energy with respect to the vacuum one has the following expansion in $\frac{1}{2} - q_0$ (which is an effective expansion in $T - T_d$)

$$\delta V = \frac{60 - d^2}{90}(1 - b)^2 \left(\frac{1}{2} - q_0\right)^2 + O\left(\left(\frac{1}{2} - q_0\right)^3\right). \tag{41}$$

The kinetic energy can be taken as at $T = T_d$, ignoring higher order corrections Eq. (27). With these ingredients, we can already work out the contribution to the interface tension from part I. It is worth noting prior doing any computations that $\delta V \sim \left(\frac{1}{2} - q_0\right)^2$, $K \sim O(1)$.

Next we consider part II. This case is particular easy because of the constant eigenvalue density in this part. To leading order, the potential energy and kinetic energy are

$$\delta V = \frac{1}{90}\left[60 - d^2 - 60 f(1 - f)\Delta(1 - \Delta)\left(12 - d^2\Delta(1 - \Delta)\right)\right]\left(\frac{1}{2} - q_0\right)^2,$$
$$T = f'^2 \Delta(1 - \Delta)\left(\frac{1}{2} - q_0\right)^2. \tag{42}$$

We note in part II both potential and kinetic terms are suppressed by $\left(\frac{1}{2} - q_0\right)^2$, leading only to contributions of higher order in $(T - T_d)$ for the interface tension.

Adding up contributions from all three parts, we obtain a contribution to leading order which is twice that of part I. It is not difficult to convince ourselves that $\alpha \sim \left(\frac{1}{2} - q_0\right)$ and it is independent of Δ.

We will improve the result by considering a more sophisticated ansatz. Note that in Eq. (39), we have chosen to turn on the b-mode and separating f-mode separately. Here we consider more general ansatz by turning them on simultaneously. Furthermore, as we learn

from the previous example that part II has only subleading contributions, we consider the abovementioned modification to part I only. With these in mind, we consider the following ansatz

$$\rho = a(z) + b(z) \cos d(q + (1 - \Delta) f_L(z)), \quad -q_0 - (1 - \Delta) f_L < q < Q_b(\Delta) - (1 - \Delta) f_L$$

$$\rho = a(z) + b(z) \cos d(q - \Delta f_R(z)), \quad Q_b(\Delta) + \Delta f_R < q < q_0 + \Delta f_R. \tag{43}$$

The tracelessness condition forces $f_L = f_R \equiv f$. It is also natural to require $f \sim 1 - 2q_0$ such that the eigenvalues do not overseparate along the path. The introduction of the f-mode induce corrections to the potential in Eq. (42). Defining $f = \bar{f} \left(\frac{1}{2} - q_0 \right) + O \left((\frac{1}{2} - q_0)^2 \right)$, we only need to consider corrections up to the second order in $\bar{f}$. It turns out the first order in $\bar{f}$ has a coefficient $P(\Delta, b)(\frac{1}{2} - q_0) + O \left((\frac{1}{2} - q_0)^2 \right)$ and the $\bar{f}^2$ term has a coefficient $Q(\Delta, b) + O \left((\frac{1}{2} - q_0) \right)$. When both are taken into account, we have for the potential

$$\delta V = \left(\delta V_0(b) + P(\Delta, b)\bar{f} + Q(\Delta, b)\bar{f}^2 \right) \left(\frac{1}{2} - q_0 \right)^2, \tag{44}$$

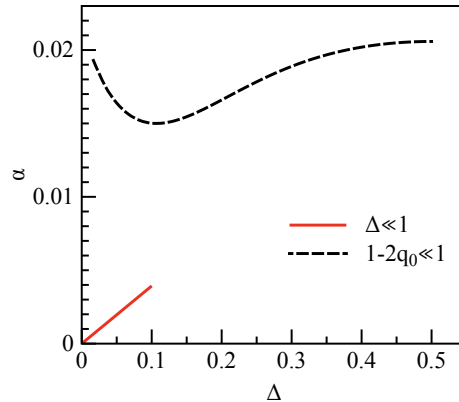FIG. 1. Interface tension at $q_0 = 0.3$. Due to the symmetry $\Delta \leftrightarrow 1 - \Delta$, we have only shown half range of Δ. The red linear line is for $\Delta \ll 1$, while the black dashed line is obtained assuming $1 - 2q_0 \ll 1$ (i.e. $T \to T_d$).

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where

$$\delta V_0 = \frac{1}{90}\left[60 - d^2 - 60f(1-f)\Delta(1-\Delta)\left(12 - d^2\Delta(1-\Delta)\right)\right], \tag{45}$$

$$P = -\frac{1}{48}(b-1)\left(32b + (1 - 4Q_b(\Delta)^2)(-48 + d^2(1 + 8Q_b(\Delta) + 12Q_b(\Delta)^2 - 16\Delta Q_b(\Delta))))\right)$$

$$- 16b(-1 + 12Q_b(\Delta)^2)\cos dQ_b(\Delta), \tag{46}$$

$$Q = 384b^2 - 48bd^2\left(1 + 4Q_b(\Delta)^2 - Q_b(\Delta)(4 - 8\Delta)\right) + d^2\Big[48 - 192Q_b(\Delta)^2$$

$$+ d^2\left(-5 - 24Q_b(\Delta)^2 + 48Q_b(\Delta)^4 + Q_b(\Delta)^3(32 - 64\Delta) + 16\Delta(1-\Delta) + 24Q_b(\Delta)(-1+2\Delta)\right)\Big]$$

$$+ 48b\left(8b + d^2(1 - 4Q_b(\Delta)^2)\right)\cos dQ_b(\Delta). \tag{47}$$

We have also factored out the overall $\left(\frac{1}{2} - q_0\right)^2$ dependence. One important property we confirm numerically is that $Q > 0$. To lower the potential energy, we choose $\bar{f} - \frac{P}{2Q}$. As a result,

$$\delta V = \left(\delta V_0 - \frac{P^2}{4Q}\right)\left(\frac{1}{2} - q_0\right)^2. \tag{48}$$

At the same time, the correction also introduces Δ dependence to the interface tension. With the same kinetic energy as at leading order, we obtain the interface tension

$$\alpha = 2\int dz\sqrt{\delta V K} \sim (\frac{1}{2} - q_0). \tag{49}$$

The final result is obtained numerically and shown in Fig. (1). Both scenarios are included.

We close this section with the observation that these different scenarios do not have a common region in which they are both valid. This suggests the limits $T \to T_d$ and $\Delta \to 0$ (or $N \to \infty$) do not commute. Therefore, it is important to evaluate corrections at finite N near T_d.

IV. FINITE N CORRECTION NEAR T_d

Corrections at finite N enter where the integrals at $N = \infty$ are replaced by discrete sums. This can be evaluated with Euler-MacLaurin formula [28]

$$\sum_{i=1}^{N} F(i) = N\int_{\frac{1}{N}}^{1} dx f(x) + \frac{f(1) + f(\frac{1}{N})}{2} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!N^{2k-1}}\left(f^{(2k-1)\prime}(1) - f^{(2k-1)\prime}\left(\frac{1}{N}\right)\right), \tag{50}$$

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where $f(i/N) = F(i)$ and B_{2k} are Bernoulli numbers. We wish to shift the argument and the lower integration bound from $\frac{1}{N}$ to 0. This reshuffle can be done with a Taylor expansion. We end up with the following surprisingly simple expression:

$$\sum_{i=1}^{N} F(i) = N \int_0^1 dx f(x) + \frac{f(1) - f(0)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)! N^{2k-1}} \left(f^{(2k-1)\prime}(1) - f^{(2k-1)\prime}(0) \right). \quad (51)$$

We are not able able to prove Eq. (51), which requires a recursion relation among Bernoulli numbers. We have verified explicitly that it holds for the first few terms in the expansion. Applying Eq. (51) to the sums, we have

$$N = \sum_i^N 1 = N \int_0^1 dx \quad (52)$$

$$0 = \sum_i^N q = N \int_0^1 dx q + \frac{q(1) - q(0)}{2} + \cdots \quad (53)$$

$$V = \sum_{i,j=1}^N V(q_i, q_j) = N^2 \int_0^1 dx dy V(q(x) - q(y)) + 2N \int_0^1 dx \frac{V(q(x) - q(1)) - V(q(x) - q(0))}{2}$$

$$+ \cdots, \quad (54)$$

where $\cdots$ denote terms higher order in $\frac{1}{N}$ expansion. As before, we define the eigenvalue density $\rho(q) = \frac{dx}{dq}$, in terms of which Eq. (52) becomes

$$1 = \int_{q_-}^{q_+} dq \rho(q) \quad (55)$$

$$0 = \int_{q_-}^{q_+} dq \rho(q) q + \frac{q_+ - q_-}{2N} + \cdots \quad (56)$$

$$V = N^2 \int_{q_-}^{q_+} dq dq' \rho(q) \rho(q') V(q - q') + N \int_{q_-}^{q_+} dq \rho(q) \left(V(q_+ - q) - V(q - q_-) \right) + \cdots, \quad (57)$$

with $q_- = q(0)$ and $q_+ = q(1)$. For the infinite N eigenvalue distribution, $\rho(q) = \rho(-q)$ and $q(0) + q(1) = 0$. The correction at next to leading order to V vanishes identically. However this need not be true for the corrected distribution in $\frac{1}{N}$.

To determine the new eigenvalue distribution, we vary V with respect to ρ, subject to the usual normalization and tracelessness constraints in Eq. (55). This gives

$$2 \int_{q_-}^{q_+} dq' \rho(q') V(q - q') + \frac{1}{N} \left(V(q_+ - q) - V(q - q_-) \right) + \lambda_1 + \lambda_2 q = 0 \quad (58a)$$

$$\int_{q_-}^{q_+} dq \rho(q) = 1 \quad (58b)$$

$$\int_{q_-}^{q_+} dq \rho(q) q + \frac{q_+ - q_-}{2N} = 0. \quad (58c)$$

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We follow the method of [8] in solving for ρ. Taking derivative with respect to q four times, we arrive at

$$\rho''(q) + d^2(\rho(q) - 1) = 0. \tag{59}$$

The general solution is given by

$$\rho(q) = 1 + b\cos dq + c\sin dq. \tag{60}$$

Plugging it into Eq. (58a), we find the result is organized as a fourth order polynomial in q. The coefficients of q and the constant term can always be set to zero by choice of the Lagrange multipliers λ_1 and λ_2. The remain coefficients of q^4, q^3 and q^2 give three independent equations. Together with Eq. (58b) and Eq. (58c), we have in total five equations, to be satisfied by four constants b, c and $q_\pm$. It turns out when four of the equations are satisfied, the fifth automatically holds. In practice, we solve for $\cos dq_\pm$ and $\sin dq_\pm$ in terms of the constants b, c and $q_\pm$. Defining $q_s = q_- + q_+$ and $q_d = q_+ - q_-$, we can first find an equation for q_d

$$\cot\frac{dq_d}{2} = -\frac{12 - d^2(1 - q_d)^2}{6d(1 - q_d)}. \tag{61}$$

It is easy to see $q_d = 2q_0$ as defined in Eq. (11c) is free of corrections in $1/N$, and that

$$q_s = \frac{1}{N}\frac{-12 - 2d^2 q_0(1 - 2q_0)^2}{d^2(1 - 2q_0)^2} \tag{62}$$

$$\frac{(144 + 12d^2(1 - 2q_0)^2 + d^4(1 - 2q_0)^4)(36 + d^2 N^2(1 - 2q_0)^4)}{d^2(1 - 2q_0)^4} = 144(b^2 + c^2)N^2 \tag{63}$$

The first shows that q_s is suppressed in $1/N$. The second combined with

$$\tan\frac{dq_s}{2} = \frac{1 - \cos d(q_- + q_+)}{\sin d(q_- + q_+)}, \tag{64}$$

can be used to determine b and c. To have a consistent $\frac{1}{N}$ expansion, we need to have

$$b = b_0 + \frac{b_2}{N^2} + \cdots, \quad c = \frac{c_1}{N} + \cdots. \tag{65}$$

At leading order b_0 agrees with Eq. (11d). c_1 is

$$c_1 = -b_0\, d\, q_0. \tag{66}$$

We have obtained Eqs. (60), (61), (62), (65) and (66) as the new eigenvalue distribution up to order $\frac{1}{N}$. The appearance of $\sin(dq)$ in Eq. (60) looks worrisome at first sight:

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the Polyakov loop can have a nonvanishing imaginary part, violating charge conjugation symmetry. The cause of the strange behavior lies in our $\frac{1}{N}$ expansion. We have taken Eq. (52) literally as the $\frac{1}{N}$ expanison and used it to solve for the eigenvalue distribution ρ to the first nontrivial order. However, the result contains constants which organize themselves in $\frac{1}{N}$ expanison, making our naive expansion inconsistent. A consistent expansion has to take into account the $\frac{1}{N}$ factors in the constants.

Using the experience from before, we expect that to order $\frac{1}{N}$, the constants have the following form:

$$b = b_0, \ c = \frac{c_1}{N}, \ q_d = 2q_0, \ q_s = \frac{q_{s1}}{N}. \tag{67}$$

Plugging Eq. (67) into Eqs. (58a), (58b), (58c) and keeping terms up to order $\frac{1}{N}$, we find, surprisingly, that the resulting equations can be solved by

$$c_1 = 0, \ q_{s1} = -\frac{12}{d^2(1 - 2q_0)^2}. \tag{68}$$

The term $\sin(d\,q)$ naturally drops out, and we find explicitly that the imaginary part of all Polyakov loops vanish to order $\frac{1}{N}$. We summarize the eigenvalue distribution as follows

$$\rho = 1 + b \cos dq, \ q_- < q < q_+ \tag{69}$$

$$q_- + q_+ = -\frac{12}{d^2 N (1 - 2q_0)^2}, \ q_+ - q_- = 2q_0. \tag{70}$$

Thus we have

$$-(q_+ - q_0)d^2 N (1 - 2q_0)^2/6 = 1. \tag{71}$$

In Fig. (2), we compare this combination with the numerical results for the maximal eigenvalue, q_{max}, which at high N is approximated by q_+, for different number of colors and
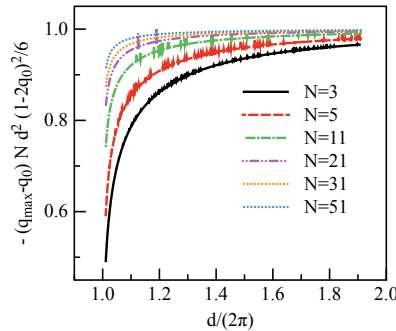

FIG. 2.   The maximal eigenvalue, q_{max}, as a function of d for different number of colors, N.

different d, that is temperature. Nice agreement between analytic expression Eq. (71) and numerical results is seen. From Eq. (69) and our numerical simulations we also see that when $N(1 - 2q_0)^2 \sim 1$, the large N expansion breaks down. We numerically computed that, at $d = 2\pi$, i.e. $T = T_d$, we have $q_{max} - q_0 \sim 1/\sqrt{N}$, in contrast to the behaviour of the maximal eigenvalue in the confined and the deconfined phase $q_{max} - q_0 \sim 1/N$.

With the correct eigenvalue distribution, we can proceed to evaluate the correction to the potential. We find the correction vanishes to order $\frac{1}{N}$. To obtain the correction to the next order, we need to find ρ at the corresponding order. We will not do it at this time, but use Eq. (69) to give an estimate of the correction. We need the following formula for the potential

$$
V = \int_{q_-}^{q_+} dq dq' \rho(q)\rho(q') V(q - q') + \frac{1}{N} \int_{q_-}^{q_+} dq \rho(q) \left(V(q_+ - q) - V(q - q_-) \right)
$$
$$
- \frac{V(q_+ - q_-)}{2N^2} + \frac{B_2}{N^2} \int_{q_-}^{q_+} dq \rho(q) \left(\frac{V'(q_+ - q)}{\rho(q_+)} + \frac{V'(q - q_-)}{\rho(q_-)} \right) + \cdots . \tag{72}
$$

Taking into account $\frac{1}{N}$ correction to q_s, we have $\frac{1}{N^2}$ contribution from all three naive orders. We will not spell out the detail of the calculation, but only list the final results:

$$
V_1 = \frac{180 - 5d^4 q0^2 (1 - 2q_0)^5 + 24d^2 q_0^2 (-5 + 20q_0 - 30q_0^2 + 16q_0^3)}{30 d^2 N^2 (1 - 2q_0)^3} \tag{73}
$$
$$
V_2 = -\frac{q_0(1 - 2q_0)(6 - d^2 q_0(1 - 2q_0))}{3N^2} \tag{74}
$$
$$
V_3 = \frac{q_0(1 - 2q_0)(6 - d^2 q_0(1 - 2q_0))}{6N^2}, \tag{75}
$$

where V_1, V_2 and V_3 denote contributions from naive order 1, $\frac{1}{N}$ and $\frac{1}{N^2}$ respectively. We note the large N expansion breaks down as $N^2 (1 - 2q_0)^3 \sim 1$. The bulk thermodynamic quantity has a less stringent criterion than the eigenvalue distribution.
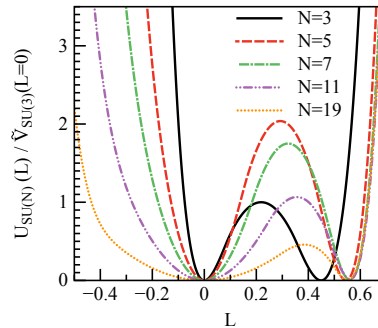

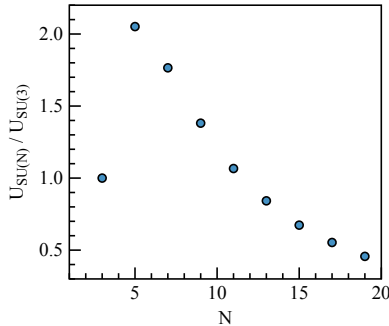FIG. 3. The non-equilibrium potential $U(L) = \tilde{V}(L) - \tilde{V}(L = 0)$.

FIG. 4. The maximum of the potential $U_{\max}$ for different N.

We were able to analytically estimate corrections to the potential at the minumum only. Numerically it is possible to go beyond this and compute the potential as a function of the Polyakov loop L. In Fig. (3), the potential normalized by $N^2 - 1$ is shown as a function of L for different number of colors. It is remarkable, that the potential flattens from the side of the confinement point $L_d = 0$. The potential maximum characterizes the order-disorder and order-order interface tensions and it falls like $1/N^2$ for large N, which is in qualitatively agreement with large N expansion, as demonstrated in Fig. (4).

V. CONCLUSIONS

The matrix model studied here is clearly only one of many possible matrix models. Its advantage is that it can be solved analytically at infinite N, and numerically at finite N. In this model all interface tensions vanish at the deconfining phase transition. Since the transition is of first order, this would be striking evidence that it is an unusual transition, perhaps of the Gross-Witten-Wadia type.

Presently, numerical simulations of $SU(N)$ gauge theories can only be carried out at relatively small N, $N < 10$. For two colors the order-disorder interface tension vanishes, as the transition is of second order. For three colors one the order-disorder interface tension is relatively small [25]. This presumably reflects that the transition for three colors is weakly first order, because of its proximity to the second order transition for two colors. This leads one to expect that as N increases, that the order-disorder interface tension, divided by N^2, increases monotonically from $N = 3$, and becomes constant at infinite N.

The numerical solution of the matrix model indicates the contrary, that the order-disorder

interface tension, divided by N^2, behaves non-monotically with N. From Fig. (4), the barrier of the potential, suitably normalized, increases from $N = 3$ to $N = 5$, and then slowly decreases as N increases further.

Such non-monotonic behavior in N is unexpected, and could well just be an artifact of the model. This could be settled by numerical studies on the lattice of the order-disorder interface tension for moderate values of $N \sim 5$. It might even provide hints of a Gross-Witten-Wadia transition at infinite N.

ACKNOWLEDGMENTS

[1] C. DeTar and U. Heller, Eur.Phys.J. **A41**, 405 (2009), arXiv:0905.2949 [hep-lat].

[2] P. Petreczky, J.Phys. **G39**, 093002 (2012), arXiv:1203.5320 [hep-lat].

[3] P. N. Meisinger, T. R. Miller, and M. C. Ogilvie, Phys.Rev. **D65**, 034009 (2002), arXiv:hep-ph/0108009 [hep-ph].

[4] P. N. Meisinger and M. C. Ogilvie, Phys.Rev. **D65**, 056013 (2002), arXiv:hep-ph/0108026 [hep-ph].

[5] A. Dumitru, Y. Guo, Y. Hidaka, C. P. Korthals Altes, and R. D. Pisarski, Phys.Rev. **D83**, 034022 (2011), arXiv:1011.3820 [hep-ph].

[6] A. Dumitru, Y. Guo, Y. Hidaka, C. P. K. Altes, and R. D. Pisarski, (2012), arXiv:1205.0137 [hep-ph].

[7] K. Kashiwa, R. D. Pisarski, and V. V. Skokov, Phys.Rev. **D85**, 114029 (2012), arXiv:1205.0545 [hep-ph].

[8] R. D. Pisarski and V. V. Skokov, Phys.Rev. **D86**, 081701 (2012), arXiv:1206.1329 [hep-th].

[9] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev.Mod.Phys. **53**, 43 (1981).

[10] S. R. Wadia, (Univ. of Chicago preprint EFI-79/44).

[11] S. R. Wadia, Phys.Lett. **B93**, 403 (1980).

[12] B. Sundborg, Nucl.Phys. **B573**, 349 (2000), arXiv:hep-th/9908001 [hep-th].

[13] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, Adv.Theor.Math.Phys. **8**, 603 (2004), arXiv:hep-th/0310285 [hep-th].

[14] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, Phys.Rev. **D71**, 125018 (2005), arXiv:hep-th/0502149 [hep-th].

[15] H. J. Schnitzer, Nucl.Phys. **B695**, 267 (2004), arXiv:hep-th/0402219 [hep-th]; L. Alvarez-Gaume, C. Gomez, H. Liu, and S. R. Wadia, Phys.Rev. **D71**, 124023 (2005), arXiv:hep-th/0502227 [hep-th]; L. Alvarez-Gaume, P. Basu, M. Marino, and S. R. Wadia, Eur.Phys.J. **C48**, 647 (2006), arXiv:hep-th/0605041 [hep-th]; T. J. Hollowood and J. C. Myers, JHEP **0911**, 008 (2009), arXiv:0907.3665 [hep-th]; S. Hands, T. J. Hollowood, and J. C. Myers, JHEP **1007**, 086 (2010), arXiv:1003.5813 [hep-th]; T. J. Hollowood, S. P. Kumar, and J. C. Myers, JHEP **1111**, 138 (2011), arXiv:1110.0696 [hep-th]; T. J. Hollowood and J. C. Myers, (2012), arXiv:1207.4605 [hep-th].

[16] A. Dumitru, J. Lenaghan, and R. D. Pisarski, Phys.Rev. **D71**, 074004 (2005), arXiv:hep-ph/0410294 [hep-ph].

[17] T. Bhattacharya, A. Gocksch, C. P. Korthals Altes, and R. D. Pisarski, Phys.Rev.Lett. **66**, 998 (1991).

[18] T. Bhattacharya, A. Gocksch, C. P. Korthals Altes, and R. D. Pisarski, Nucl.Phys. **B383**, 497 (1992), arXiv:hep-ph/9205231 [hep-ph].

[19] P. Giovannangeli and C. P. Korthals Altes, Nucl.Phys. **B721**, 1 (2005), arXiv:hep-ph/0212298 [hep-ph].

[20] P. Giovannangeli and C. P. Korthals Altes, Nucl.Phys. **B721**, 25 (2005), arXiv:hep-ph/0412322 [hep-ph].

[21] S. A. Hartnoll and S. P. Kumar, Phys.Rev. **D74**, 026001 (2006), arXiv:hep-th/0603190 [hep-th].

[22] A. Armoni, S. P. Kumar, and J. M. Ridgway, JHEP **0901**, 076 (2009), arXiv:0812.0773 [hep-th].

[23] C. P. Korthals Altes, A. Kovner, and M. A. Stephanov, Phys.Lett. **B469**, 205 (1999), arXiv:hep-ph/9909516 [hep-ph].

[24] C. P. Korthals Altes and A. Kovner, Phys.Rev. **D62**, 096008 (2000), arXiv:hep-ph/0004052 [hep-ph].

[25] B. Beinlich, F. Karsch, and A. Peikert, Phys.Lett. **B390**, 268 (1997), arXiv:hep-lat/9608141 [hep-lat].

[26] B. Lucini, M. Teper, and U. Wenger, JHEP **0502**, 033 (2005), arXiv:hep-lat/0502003 [hep-lat].

[27] J. Polchinski, Phys.Rev.Lett. **68**, 1267 (1992), arXiv:hep-th/9109007 [hep-th].

[28] Y. Y. Goldschmidt, J.Math.Phys. **21**, 1842 (1980).